

INJECTIVE STABILIZATION OF ADDITIVE FUNCTORS. III. ASYMPTOTIC STABILIZATION OF THE TENSOR PRODUCT

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ABSTRACT. The injective stabilization of the tensor product is subjected to an iterative procedure that utilizes its bifunctor property. The limit of this procedure, called the asymptotic stabilization of the tensor product, provides a homological counterpart of Buchweitz's asymptotic construction of stable cohomology. The resulting connected sequence of functors is isomorphic to Triulzi's J -completion of the Tor functor. A comparison map from Vogel homology to the asymptotic stabilization of the tensor product is constructed and shown to be always epic.

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1. STABLE COHOMOLOGY

Around 1950, John Tate noticed that the trivial module \mathbb{k} over the group ring $\mathbb{k}G$ (where \mathbb{k} is a field or the ring of rational integers) of a finite group G admits a

projective coresolution. Splicing it together with a projective resolution of the same module, he obtained a doubly infinite exact complex of projectives, called a complete resolution of \mathbb{k} . Using it in place of the projective resolution of \mathbb{k} , he modified the usual notion of group cohomology, obtaining what is now known as Tate cohomology. For more details, the reader is referred to [1] and [3].

In 1977, F. T. Farrell [4] constructed a cohomology theory for groups of finite virtual cohomological dimension that, for finite groups, gave the same result as Tate cohomology.

In the mid-1980s, R.-O. Buchweitz [2] constructed a generalization of Tate (and Farrell) cohomology that worked over arbitrary Gorenstein rings.

1.1. Vogel cohomology. At about the same time, Pierre Vogel [5] came up with his own generalization of Tate cohomology, and while he was interested in arbitrary group rings, his approach actually worked over any ring. We now review that construction.

Let Λ be a (unital) ring and M and N (left) Λ -modules. Choose projective resolutions $(\mathbf{P}, \partial) \rightarrow M$ and $(\mathbf{Q}, \partial) \rightarrow N$. Forgetting the differentials, we have \mathbb{Z} -diagrams \mathbf{P} and \mathbf{Q} of left Λ -modules, together with a \mathbb{Z} -diagram (\mathbf{P}, \mathbf{Q}) of abelian groups. The latter has $\prod_i \text{Hom}(P_i, Q_{i+n})$ as its degree n component. It contains the subdiagram $(\mathbf{P}, \mathbf{Q})_b$ of bounded maps, whose degree n component is $\prod_i \text{Hom}(P_i, Q_{i+n})$. Passing to the quotient, we have a short exact sequence of diagrams

$$0 \rightarrow (\mathbf{P}, \mathbf{Q})_b \rightarrow (\mathbf{P}, \mathbf{Q}) \rightarrow (\widehat{\mathbf{P}}, \widehat{\mathbf{Q}}) \rightarrow 0$$

The usual formula, $D(f) := \partial \circ f - (-1)^{\deg f} f \circ \partial$, defines a differential on the middle diagram, which clearly restricts to a differential on the subdiagram of bounded maps. Thus the inclusion map is actually an inclusion of complexes, and the corresponding quotient becomes the quotient complex. By construction, the maps in this short exact sequence are chain maps between the constructed complexes. The n th Vogel cohomology of M with coefficients in N , where $n \in \mathbb{Z}$, is then defined as the n th cohomology of the complex $(\widehat{\mathbf{P}}, \widehat{\mathbf{Q}})$. We denote it by $V^n(M, N)$.

1.2. Buchweitz cohomology. As we mentioned before, Buchweitz was interested in a generalized Tate cohomology over Gorenstein rings, but his construction (actually, one of two proposed) turned out to work for any ring. We now describe his approach. Again, let Λ be an arbitrary (unital) ring, M and N (left) Λ -modules, and $\Lambda\text{-Mod}$ the category of left Λ -modules and homomorphisms. First, we pass to the category $\Lambda\text{-}\underline{\text{Mod}}$ of modules modulo projectives, which has the same objects as $\Lambda\text{-Mod}$, but whose morphisms $\underline{(M, N)}$ are defined as the quotient groups $(M, N)/P(M, N)$, where $P(M, N)$ is the subgroup of all maps that can be factored through a projective module. The composition of classes of homomorphisms is defined as the class of the composition of representatives. One of the advantages of this new category is that the syzygy operation Ω on $\Lambda\text{-Mod}$ becomes an additive endofunctor on $\Lambda\text{-}\underline{\text{Mod}}$. In particular, for M and N we have a sequence of homomorphisms of abelian groups

$$\underline{(M, N)} \rightarrow (\underline{\Omega M}, \underline{\Omega N}) \rightarrow (\underline{\Omega^2 M}, \underline{\Omega^2 N}) \rightarrow \dots$$

The n th Buchweitz cohomology $B^n(M, N)$, $n \in \mathbb{Z}$ is defined as

$$\varinjlim_{n+k, k \geq 0} (\Omega^{n+k} M, \Omega^k N).$$

1.3. Mislin's construction. Yet another generalization of Tate cohomology was given by G. Mislin [7] in 1994. It came as a special case of a considerably more general construct. For a cohomological (or, more generally, connected) sequence of functors $\{F^i\}$, $i \in \mathbb{Z}$ Mislin constructs a sequence of natural transformations

$$F^i \longrightarrow S_1(F^{i+1}) \longrightarrow S_2(F^{i+2}) \longrightarrow \dots,$$

where S_j denotes the j th left satellite, and defines what he calls the P -completion of $\{F^i\}$ as

$$\varinjlim_{k \geq 0} S_k(F^{i+k}) =: M^i F.$$

Evaluating the colimit on the group cohomology (viewed as a cohomological functor of the coefficients), he gets a new cohomological (or connected if the original sequence is connected but not necessarily cohomological) sequence of functors. He then proves that, for groups of finite virtual cohomological dimension, the new cohomology is isomorphic to Farrell cohomology. Moreover, he also establishes, for arbitrary groups, an isomorphism between his construction and Buchweitz's cohomology (called in the paper the Benson-Carlson cohomology, after the two authors, who independently found Buchweitz's cohomology in 1992). It should be clear, however, that Mislin's construction is completely general and applies, in particular, to the Ext functor over any ring.

2. STABLE HOMOLOGY

At this point, one may ask if there are **homological** analogs of the various cohomology theories discussed above. The answer to this question is less clear. First, there was no "Tate homology" in Tate's original work: only the Hom functor was used with complete resolutions. However, at the same time when P. Vogel constructed his cohomology, he also constructed a homology theory. We begin by reviewing his construction.

2.1. Vogel homology. Let Λ be a ring, M a left Λ -module and N a right Λ -module. Choose a projective resolution $(\mathbf{P}, \partial) \longrightarrow M$ and an injective resolution $N \longrightarrow (\mathbf{I}, \partial)$. Forgetting the differentials, we have \mathbb{Z} -diagrams \mathbf{P} and \mathbf{I} of left and, respectively, right Λ -modules, together with a \mathbb{Z} -diagram $\mathbf{P} \hat{\otimes} \mathbf{I}$ of abelian groups. The latter has $\prod_i (P_i \otimes I^{i-n})$ as its degree n component. It contains the subdiagram $\mathbf{P} \otimes \mathbf{I}$, whose degree n component is $\prod_i (P_i \otimes I^{i-n})$. Passing to the quotient, we have a short exact sequence of diagrams

$$0 \longrightarrow \mathbf{P} \otimes \mathbf{I} \longrightarrow \mathbf{P} \hat{\otimes} \mathbf{I} \longrightarrow \mathbf{P} \overset{\vee}{\otimes} \mathbf{I} \longrightarrow 0$$

The standard definition

$$\begin{aligned} (2.1) \quad D(a \otimes b) &:= \partial_P(a) \otimes b + (-1)^{\deg a} a \otimes \partial_I(b) \\ &= (\partial_P \otimes 1 + (-1)^{\deg_1(-)} 1 \otimes \partial_I)(a \otimes b), \end{aligned}$$

where a and b are homogeneous elements of \mathbf{P} and, respectively, \mathbf{I} , and $\deg_1(-)$ picks the degree of the first factor of a decomposable tensor, gives rise to a differential on $\mathbf{P} \otimes \mathbf{I}$. It is not difficult to see that it extends to a differential, denoted by D again, on $\mathbf{P} \hat{\otimes} \mathbf{I}$. Indeed, if $s \in (\mathbf{P} \hat{\otimes} \mathbf{I})_n$ is a degree n element, then $s = (s_i)_{i \in \mathbb{Z}}$,

where each $s_i \in P_i \otimes I^{i-n}$ is just a finite sum of decomposable tensors. For each $k \in \mathbb{Z}$, define

$$D : \prod_i (P_i \otimes I^{i-n}) \longrightarrow (P_k \otimes I^{k+1-n}) : s \mapsto (\partial \otimes 1)(s_{k+1}) + (-1)^k (1 \otimes \partial)(s_k)$$

Now, we obtain the desired differential by the universal property of direct product.

As a consequence, the third term in the short exact sequence above becomes a complex, and Vogel homology is now defined by setting

$$(2.2) \quad V_n(M, N) := H_{n+1}(\mathbf{P} \overset{\vee}{\otimes} \mathbf{I}).$$

Remark 2.1.1. Because of the shift in the subscript, the connecting homomorphism in the long homology exact sequence is a map $V_n(M, N) \longrightarrow \text{Tor}_n(M, N)$.

Remark 2.1.2. The choice of the projective and injective resolutions can be flipped. By choosing an injective resolution of M and a projective resolution of N , one obtains another homological functor, which in general is different from the original one. This can be seen by choosing M to be projective. In that case, the original functor evaluates to zero, whereas the alternative construction produces, in general, a nonzero result.

2.2. A homological analog of Mislin's construction. A homological analog of Mislin's cohomological P -completion, called the J -completion, was defined by M. Triulzi in his PhD thesis [9]¹. Like its cohomological prototype, it is defined on connected sequences of functors, but even if the original sequence is cohomological, the result doesn't seem to be cohomological²; one can only claim that the resulting sequence is connected. For reference, we denote it by $M_i F$.

2.3. Summary. We summarize the existing constructions in the following table:

Cohomology	Homology
$V^i(M, N)$	$V_i(M, N)$
$B^i(M, N)$?
$M^i F$	$M_i F$

One of the goals of this paper is to replace the question mark by a homological analog of Buchweitz's construction.

In this paper, we follow the terminology and notation established in [6]. The reader may benefit from reviewing that source.

Some results contained in the present paper overlap with some results obtained by the second author in his PhD thesis [8].

3. THE ASYMPTOTIC STABILIZATION OF THE TENSOR PRODUCT

Our next goal is to introduce what we shall call the **asymptotic stabilization** of the tensor product, which is a limit of a sequence of maps between injective stabilizations of tensor products of iterated syzygy and cosyzygy modules. In this section, this will be done in three equivalent ways.

¹The authors are grateful to Lucho Avramov for bringing this work to our attention and to Lars Christensen for sending us a copy of it

²This is related to the fact that the inverse limit is not an exact functor.

Blanket assumption. Whenever we deal with a connecting homomorphism in the snake lemma, we automatically assume that the homomorphism was constructed by pushing and pulling the elements along a staircase path, as in the traditional proof of the lemma.

3.1. The first construction. We begin with constructing a homomorphism

$$\Omega A \overset{\sim}{\otimes} \Sigma B \longrightarrow A \overset{\sim}{\otimes} B$$

of abelian groups, where A is a right Λ -module and B is a left Λ -module.

Given a left Λ -module B , choose an injective resolution

$$(3.1) \quad 0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

Similarly, given a right Λ -module A choose a projective resolution

$$(3.2) \quad \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

Tensoring the short exact sequences

$$0 \longrightarrow \Omega A \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow B \longrightarrow I^0 \longrightarrow \Sigma B \longrightarrow 0$$

we have the following commutative diagram of solid arrows whose rows, columns, and diagonal are exact:

$$(3.3) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \searrow & & \downarrow & & \\ & & \Omega A \overset{\sim}{\otimes} \Sigma B & \xrightarrow{\quad \quad} & \text{Tor}_1(A, \Sigma B) & & \\ & & \searrow & & \downarrow & & \\ \Omega A \otimes B & \longrightarrow & \Omega A \otimes I^0 & \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \searrow \\ 0 \longrightarrow & P_0 \otimes B & \longrightarrow & P_0 \otimes I^0 & \longrightarrow & P_0 \otimes \Sigma B & \longrightarrow \Omega A \otimes I^1 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \dots \longrightarrow & A \otimes B & \longrightarrow & A \otimes I^0 & \longrightarrow & A \otimes \Sigma B & \longrightarrow P_0 \otimes I^1 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ & 0 & & 0 & & 0 & \end{array}$$

Using the fact that $P_0 \otimes _$ is an exact functor and the snake lemma, we have

Lemma 3.1.1. *The above solid diagram induces an exact sequence*

$$\text{Tor}_1(A, B) \longrightarrow \text{Tor}_1(A, I^0) \longrightarrow \text{Tor}_1(A, \Sigma B) \xrightarrow{\delta} A \overset{\sim}{\otimes} B \longrightarrow 0,$$

where δ is (the corestriction of) the connecting homomorphism. If the injective I^0 is projective, then δ is an isomorphism. \square

As $P_0 \otimes _$ is an exact functor, the bottom southeast map is monic. The composition of this map with

$$\Omega A \overset{\sim}{\otimes} \Sigma B \longrightarrow \Omega A \otimes \Sigma B \longrightarrow P_0 \otimes \Sigma B$$

is obviously zero and, by the universal property of kernels, we have the dotted map in the above diagram making the top triangle commute. Notice that this map is monic. Applying the snake lemma yields the following diagram with an exact bottom row

$$\begin{array}{c} \Omega A \overset{\overline{\otimes}}{\otimes} \Sigma B \\ \downarrow \\ \text{Tor}_1(A, \Sigma B) \longrightarrow A \otimes B \longrightarrow A \otimes I^0 \end{array}$$

Because the composition of the bottom maps is zero, this diagram embeds in the commutative diagram

$$\begin{array}{ccccccc} & & \Omega A \overset{\overline{\otimes}}{\otimes} \Sigma B & & & & \\ & & \downarrow & & & & \\ & & \text{Tor}_1(A, \Sigma B) & \longrightarrow & A \otimes B & \longrightarrow & A \otimes I^0 \\ & & \downarrow \delta & & \parallel & & \parallel \\ 0 & \longrightarrow & A \overset{\overline{\otimes}}{\otimes} B & \longrightarrow & A \otimes B & \longrightarrow & A \otimes I^0 \end{array}$$

which produces a homomorphism

$$\Omega A \overset{\overline{\otimes}}{\otimes} \Sigma B \xrightarrow{\Delta_1} A \overset{\overline{\otimes}}{\otimes} B$$

Iteration of this process yields a directed system

$$(3.4) \quad \dots \longrightarrow \Omega^2 A \overset{\overline{\otimes}}{\otimes} \Sigma^2 B \xrightarrow{\Delta_2} \Omega A \overset{\overline{\otimes}}{\otimes} \Sigma B \xrightarrow{\Delta_1} A \overset{\overline{\otimes}}{\otimes} B.$$

Now we want to show that any two choices for Δ_1 , and hence for any other Δ_i , are isomorphic. In addition to the resolutions (3.1) and (3.2), let

$$(3.5) \quad 0 \longrightarrow B \longrightarrow I^{0'} \longrightarrow I^{1'} \longrightarrow \dots$$

be another injective resolution of B , and

$$(3.6) \quad \dots \longrightarrow P_1' \longrightarrow P_0' \longrightarrow A \longrightarrow 0$$

another projective resolution of A . Lifting the identity map on A , extending the identity map on B , and taking the tensor product results in a commutative 3D version of diagram (3.3). By the naturality of the connecting homomorphism in

the snake lemma, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathrm{Tor}_1(A, \Sigma' B) & \longrightarrow & A \otimes B & \longrightarrow & A \otimes I^{0'} \longrightarrow \dots \\
 & & \searrow & & \parallel & & \downarrow \\
 & & & A \otimes B & & & \\
 & & \alpha \cong & & & & \\
 \dots & \longrightarrow & \mathrm{Tor}_1(A, \Sigma B) & \longrightarrow & A \otimes B & \longrightarrow & A \otimes I^0 \longrightarrow \dots \\
 & & \searrow & & \parallel & & \downarrow \\
 & & & A \otimes B & & &
 \end{array}$$

By [6, Lemma 3.2)], α is an equality. On the other hand, the right-hand side of the 3D-version of (3.3) yields a commutative diagram of solid arrows

$$\begin{array}{ccccccc}
 & & & & \mathrm{Tor}_1(A, \Sigma' B) & & \\
 & & & & \downarrow & & \\
 & & & \mathrm{Tor}_1(A, \Sigma B) & & & \\
 & & & \downarrow & & & \\
 0 \longrightarrow & \Omega' A \otimes \Sigma' B & \longrightarrow & \Omega' A \otimes \Sigma' B & \longrightarrow & \Omega' A \otimes I^{1'} & \\
 & \cong \searrow \beta & & \downarrow & & \downarrow & \\
 0 \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & \Omega A \otimes I^1 & \\
 & & & \downarrow & & \downarrow & \\
 & & & P_0' \otimes \Sigma' B & \longrightarrow & P_0' \otimes I^{1'} & \\
 & & & \downarrow & & \downarrow & \\
 & & & P_0 \otimes \Sigma B & \longrightarrow & P_0 \otimes I^1 &
 \end{array}$$

with exact rows and columns. The dotted arrows also come from diagram (3.3) and make the triangles containing them commute. By [6, Lemmas 8.1 and 8.2], β is the canonical isomorphism. Using the fact that the map $\mathrm{Tor}_1(A, \Sigma B) \rightarrow \Omega A \otimes \Sigma B$ is a monomorphism, we have that the curved square also commutes. Splicing it with the left-hand square containing α from the preceding diagram, we have a commutative square

$$\begin{array}{ccc}
 \Omega' A \otimes \Sigma' B & \xrightarrow{\Delta'_1} & A \otimes B \\
 \cong \beta \downarrow & & \cong \alpha \downarrow \\
 \Omega A \otimes \Sigma B & \xrightarrow{\Delta_1} & A \otimes B
 \end{array}$$

with the vertical maps being canonical isomorphisms. This proves

Proposition 3.1.2. *Any two choices for Δ_1 , and hence for any Δ_i , based on the diagram (3.3) are canonically isomorphic.* \square

Arguments very similar to the ones just used yield

Proposition 3.1.3. *The homomorphism $\Delta_1 : \Omega A \overset{\rightharpoonup}{\otimes} \Sigma B \longrightarrow A \overset{\rightharpoonup}{\otimes} B$, and hence any Δ_i , is functorial in both A and B . \square*

For any integer $n \in \mathbb{Z}$ (including negative values), the process of constructing the directed system (3.4) may be repeated with $\Omega^{k+n} A$ in place of $\Omega^k A$, yielding directed systems

$$(3.7) \quad M_n(A, B) := \Omega^{k+n} A \overset{\rightharpoonup}{\otimes} \Sigma^k B, \quad k, k+n \geq 0$$

Definition 3.1.4. *The asymptotic stabilization $T_n(A, _)$ of the left tensor product in degree n with coefficients in the right Λ -module A is*

$$(3.8) \quad \begin{aligned} T_n(A, _)(B) &:= T_n(A, B) \\ &:= \varprojlim_{k, k+n \geq 0} \Omega^{k+n} A \overset{\rightharpoonup}{\otimes} \Sigma^k B = \varprojlim M_n(A, B) \end{aligned}$$

It is easy to see that the $T_n(A, _) : \Lambda\text{-Mod} \longrightarrow \text{Ab}$, $n \in \mathbb{Z}$ are covariant additive functors from the category of left Λ -modules to the category of abelian groups. It is plain that the $T_n(A, _)$ are injectively stable. The next result shows that we also have dimension shifts, including the fixed argument.

Lemma 3.1.5. *For all $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, and $j \in \mathbb{Z}_{\geq 0}$ there are canonical isomorphisms of functors*

$$T_n(A, \Sigma^k _) \cong T_{n-k}(A, _)$$

and

$$T_n(\Omega^j A, _) \cong T_{n+j}(A, _)$$

Proof. The directed systems (including the structure maps) for the components of the former (respectively, latter) pair of functors at any right Λ -module can be obviously chosen to be shifts of each other. \square

Now we want to discuss the vanishing of the functors $T_\bullet(A, _)$. The first result is an immediate consequence of the definitions.

Proposition 3.1.6. *If the right global dimension of Λ is finite then $T_n(A, _) = 0$ for all integers n . \square*

Proposition 3.1.7. *If the flat dimension of A is finite, then $T_n(A, _) = 0$ for all integers n .*

Proof. As the diagram (3.3) shows, we have an injection $\Omega A \overset{\rightharpoonup}{\otimes} \Sigma B \longrightarrow \text{Tor}_1(A, \Sigma B)$. In particular, $\Omega^{n+k} A \overset{\rightharpoonup}{\otimes} \Sigma^k B$, $n+k, k \geq 1$ embeds in $\text{Tor}_1(\Omega^{n+k-1} A, \Sigma^k B)$. But the latter vanishes for $n+k-1 \geq \text{fl. dim } A$. \square

It is known that the vanishing of stable cohomology in one degree implies its vanishing in all degrees. We do not know if a similar statement is true for $T_\bullet(A, _)$. A partial answer is provided by

Proposition 3.1.8. *If $T_n(A, _) = 0$ for some integer n , then $T_m(A, _) = 0$ for all $m < n$. If, in addition, Λ is quasi-Frobenius, then $T_m(A, _) = 0$ for all $m \in \mathbb{Z}$.*

Proof. The first assertion is an immediate consequence of the first isomorphism of Lemma 3.1.5. Suppose now that Λ is quasi-Frobenius. Since projective modules

are injective, for any positive integer k , any right Λ -module B is a k th cosyzygy module in an injective resolution of $\Omega^k B$, i.e., $B \simeq \Sigma^k \Omega^k B$. Therefore,

$$T_{n+k}(A, B) \cong T_{n+k}(A, \Sigma^k \Omega^k B) \cong T_n(A, \Omega^k B) = 0.$$

□

3.2. The second construction. Next we want to show that Proposition 3.1.7 follows from a more general result, namely, that the asymptotic stabilization $T_\bullet(A, B)$ can be computed via the Tor functors. Our goal is to construct a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Tor}_1(\Omega A, \Sigma^2 B) & \longrightarrow & \text{Tor}_1(A, \Sigma B) & \longrightarrow & A \otimes B \\ & & \nearrow & & \searrow & & \nearrow \\ & \dots & \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & A \otimes B & \end{array}$$

where the bottom sequence is given by (3.4), and the arrows in the top sequence are connecting homomorphisms. Clearly, once such a diagram has been constructed, the limits of the horizontal rows will be isomorphic, showing that the asymptotic stabilization can indeed be constructed using the Tor functors. Moreover, we shall also show that all northeast arrows are monic and all southeast arrows are epic. This immediately implies that all stages in the original construction of the asymptotic stabilization can be recovered via the epi-mono factorizations of the top arrows.

The construction requires explicit choices, so for a right Λ -module A we choose a projective resolution

$$\dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and use the definition of $\text{Tor}_1(A, _)$ via the exact sequence

$$(3.9) \quad 0 \longrightarrow \text{Tor}_1(A, _) \longrightarrow \Omega A \otimes _ \longrightarrow P_0 \otimes _ \longrightarrow A \otimes _ \longrightarrow 0.$$

For a projective resolution of ΩA we choose the projective resolution of A truncated in degree 1. This allows us to claim that $\text{Tor}_{i+1}(A, _) = \text{Tor}_i(\Omega A, _)$, where we do mean an equality rather than an abstract isomorphism.

For a short exact sequence

$$(3.10) \quad 0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

of left A -modules, recall the construction the connecting homomorphisms

$$\text{Tor}_{i+1}(A, E) \longrightarrow \text{Tor}_i(A, C)$$

in the corresponding long exact sequence of the Tor functors. The case $i = 0$ consists of evaluating the sequence (3.9) on the short exact sequence above and then using the snake lemma. For positive values of i , we describe the construction when $i = 1$ and then use the dimension shift. To this end, we replace A with ΩA and build a snake diagram as in the case $i = 0$. The new diagram and the original one have a common row,

$$\Omega A \otimes C \longrightarrow \Omega A \otimes D \longrightarrow \Omega A \otimes E \longrightarrow 0,$$

which allows to glue the two diagrams together:

$$\begin{array}{ccccc}
 & & \text{Tor}_1(\Omega A, C) & \longrightarrow & \text{Tor}_1(\Omega A, D) & \longrightarrow & \text{Tor}_1(\Omega A, E) \\
 & & \searrow & & \searrow & & \searrow \\
 & & \Omega^2 A \otimes C & \longrightarrow & \Omega^2 A \otimes D & \longrightarrow & \Omega^2 A \otimes E \\
 & & \searrow & & \searrow & & \searrow \\
 & & P_1 \otimes C & \longrightarrow & P_1 \otimes D & \longrightarrow & P_1 \otimes E \\
 & & \searrow & & \searrow & & \searrow \\
 \text{Tor}_1(A, C) & \longrightarrow & \text{Tor}_1(A, D) & \longrightarrow & \text{Tor}_1(A, E) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega A \otimes C & \xrightarrow{\gamma} & \Omega A \otimes D & \longrightarrow & \Omega A \otimes E \\
 \downarrow \alpha & \begin{array}{c} T \\ \downarrow \delta \end{array} & \downarrow \delta & & \downarrow \\
 P_0 \otimes C & \xrightarrow{\beta} & P_0 \otimes D & \longrightarrow & P_0 \otimes E \\
 \downarrow & & \downarrow & & \downarrow \\
 A \otimes C & \longrightarrow & A \otimes D & \longrightarrow & A \otimes E
 \end{array}$$

Notice that the connecting homomorphism

$$\text{Tor}_2(A, E) = \text{Tor}_1(\Omega A, E) \xrightarrow{\epsilon} \Omega A \otimes C$$

in the horizontal part of the diagram factors through $\text{Ker } \alpha = \text{Tor}_1(A, C)$. Indeed, the commutativity of the square T shows that $\beta\alpha\epsilon = \delta\gamma\epsilon = 0$. Since β is monic, $\alpha\epsilon = 0$ and ϵ factors through $\text{Tor}_1(A, C)$. As a result, we have a connecting homomorphism $\text{Tor}_2(A, E) \longrightarrow \text{Tor}_1(A, C)$ and the desired long exact sequence.

Returning to the left Λ -module B , we specialize the short exact sequence (3.10) to the cosyzygy sequence

$$(3.11) \quad 0 \longrightarrow \Sigma B \longrightarrow I^1 \longrightarrow \Sigma^2 B \longrightarrow 0.$$

The foregoing argument then yields a commutative square

$$\begin{array}{ccc}
 \text{Tor}_2(A, \Sigma^2 B) & \longrightarrow & \text{Tor}_1(A, \Sigma B) \\
 \parallel & \searrow & \downarrow \\
 \text{Tor}_1(\Omega A, \Sigma^2 B) & \longrightarrow & \Omega A \otimes \Sigma B
 \end{array}$$

where the diagonal map is the connecting homomorphism in the horizontal part of the diagram on page 10. The composition of this map with $\gamma : \Omega A \otimes \Sigma B \rightarrow \Omega A \otimes I^1$ is zero, hence it factors through the kernel of γ , which is by definition $\Omega A \overline{\otimes} \Sigma B$.

We now have a commutative diagram of solid arrows

$$\begin{array}{ccccc}
 \mathrm{Tor}_2(A, \Sigma^2 B) & \xrightarrow{\quad} & \mathrm{Tor}_1(A, \Sigma B) & & \\
 \parallel & \searrow & \downarrow & \nearrow \text{dotted} & \\
 & & \Omega A \bar{\otimes} \Sigma B & & \\
 \mathrm{Tor}_1(\Omega A, \Sigma^2 B) & \xrightarrow{\quad} & \Omega A \bar{\otimes} \Sigma B & \xrightarrow{\quad} & \Omega A \bar{\otimes} I^1 \\
 & & \downarrow & \xrightarrow{T} & \downarrow \\
 & & P_0 \bar{\otimes} \Sigma B & \xrightarrow{\quad} & P_0 \bar{\otimes} I^1
 \end{array}$$

The existence of the dotted arrow and the fact that it is monic was established when we discussed the diagram (3.3); it makes the triangle on the right commute. Since the vertical map in that triangle is monic, the top triangle is also commutative. By construction, the horizontal map in that triangle is the connecting homomorphism in the long exact sequence of the functors $\mathrm{Tor}_i(A, _)$ corresponding to the cosyzygy sequence (3.11). Colloquially, the connecting homomorphism in the long exact Tor-sequence factors through the injective stabilization. We view these connecting homomorphisms as the structure maps in the directed system

$$\dots \longrightarrow \mathrm{Tor}_1(\Omega^i A, \Sigma^{i+1} B) \longrightarrow \mathrm{Tor}_1(\Omega^{i-1} A, \Sigma^i B) \longrightarrow \dots$$

On the other hand, as (3.3) showed, the structure map $\Omega A \bar{\otimes} \Sigma B \longrightarrow A \bar{\otimes} B$ factors through $\mathrm{Tor}_1(A, \Sigma B)$. Combining these two observations, we have a commutative diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathrm{Tor}_1(\Omega A, \Sigma^2 B) & \longrightarrow & \mathrm{Tor}_1(A, \Sigma B) & \longrightarrow & A \bar{\otimes} B \\
 & \nearrow & \searrow & \nearrow & \searrow & & \nearrow \\
 (3.12) & & & & & & \\
 \dots & \longrightarrow & \Omega A \bar{\otimes} \Sigma B & \longrightarrow & A \bar{\otimes} B & &
 \end{array}$$

“intertwining” the two directed systems. Taking into account the dimension shift, we now have

Theorem 3.2.1. *The directed system of the Tor functors in the above diagram is functorial in both arguments. For any integer n , the two families of parallel arrows in the (suitably shifted) above diagram induce mutually inverse isomorphisms³*

$$\begin{aligned}
 T_n(A, _)(B) &= \varprojlim_{k, k+n \geq 0} \Omega^{k+n} A \bar{\otimes} \Sigma^k B \\
 &\simeq \varprojlim_{k, k+n \geq 0} \mathrm{Tor}_1(\Omega^{k+n} A, \Sigma^{k+1} B)
 \end{aligned}$$

Proof. The first assertion follows from the functoriality of the connecting homomorphism. The second assertion has already been established. \square

Remark 3.2.2. In the above diagram, A and B can be replaced by their arbitrary syzygy and, respectively, cosyzygy modules. With each map in the directed system, the powers of syzygy and cosyzygy modules simultaneously go down by one. If the

³The reader has probably noticed that this theorem implies Proposition 3.1.7.

powers of Ω run out first, then the last term on the right will be a tensor product. If the powers of Σ run out first, then the last term will be a Tor_1 .

Remark 3.2.3. We have actually proved more. Since all southeast maps are epic, all northeast maps are monic, and since an epi-mono factorization of a morphism in an abelian category is determined uniquely up to an isomorphism, the lower directed system is determined uniquely up to an isomorphism by the maps in the upper system. In particular, this yields new equivalent definitions of both the injective stabilization and the asymptotic stabilization of the tensor product.

Example 3.2.4. Suppose Λ is quasi-Frobenius. Then, by Lemma 3.1.1, the southeast maps are all isomorphisms, making the two systems isomorphic. The next example shows that the two systems may be isomorphic over other types of rings.

Example 3.2.5. Let $\Lambda := \mathbb{Z}$, $A := \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, and $B := \mathbb{Z}$. Then, taking $\Sigma B \simeq \mathbb{Q}/\mathbb{Z}$, we have, by [6, Lemma 3.5] (or, by [6, Lemma 3.8]), $\Omega A \otimes \Sigma B = 0$. To compute $A \otimes B$, we apply the functor $\mathbb{Z}/p\mathbb{Z} \otimes -$ to the injective envelope $\mathbb{Z} \rightarrow \mathbb{Q}$ of \mathbb{Z} . The kernel of the resulting map $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Q}$ is $\mathbb{Z}/p\mathbb{Z}$. Finally, we compute $\text{Tor}_1(A, \Sigma B)$ by using a projective resolution of $A = \mathbb{Z}/p\mathbb{Z}$. The result is the kernel of the map $\mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Q}/\mathbb{Z}$, which is the subgroup $\mathbb{Z}/p\mathbb{Z} = \{0, 1/p, \dots, (p-1)/p\}$. Moreover, the diagram (3.3) shows that, in this case, the map $\text{Tor}_1(A, \Sigma B) \rightarrow A \otimes B$ is an isomorphism. Since all the remaining terms in the two directed systems vanish, we have that the southeast maps in (3.12) make the two systems isomorphic.

3.3. The third construction. Recall that the right-hand side of the functorial isomorphism $S^1\text{Tor}_1(A, B) \cong A \otimes B$ in [6, Proposition 8.3] is the initial term of the directed system (3.4):

$$\dots \longrightarrow \Omega^2 A \otimes \Sigma^2 B \xrightarrow{\Delta_2} \Omega A \otimes \Sigma B \xrightarrow{\Delta_1} A \otimes B.$$

Our next goal is to construct an isomorphic directed system starting with the left-hand side. Of course, this simply means constructing structure maps. First, we need yet another observation about connecting homomorphisms. In the diagram (3.3), we have two copies of $\text{Tor}_1(A, \Sigma B)$, one at the top of the rightmost vertical exact sequence, the other (not shown) as the next term in the long exact sequence in the bottom row. Both map into $A \otimes B$, and we wish to make a commutative triangle by constructing an isomorphism between the two copies of Tor . This will be a general observation. More precisely, let

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \longrightarrow P \longrightarrow A \longrightarrow 0$$

be short exact sequences with P projective. Tensoring them together, we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & \text{Tor}_1(A, D) & & \\
& & & & \downarrow & & \\
& & \Omega A \otimes B & \longrightarrow & \Omega A \otimes C & \longrightarrow & F \otimes D \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
(3.13) \quad & 0 \longrightarrow & P \otimes B & \longrightarrow & P \otimes C & \longrightarrow & P \otimes D \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& \text{Tor}_1(A, D) \xrightarrow{\alpha} & A \otimes B & \xrightarrow{\beta} & A \otimes C & \longrightarrow & A \otimes D \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the bottom row and the rightmost column are fragments of the corresponding long homology exact sequences.

Lemma 3.3.1. *Let $\delta : \text{Tor}_1(A, D) \rightarrow A \otimes B$ be the connecting homomorphism in the above diagram. Then there is an isomorphism*

$$\gamma : \text{Tor}_1(A, D) \longrightarrow \text{Tor}_1(A, D)$$

such that $\alpha\gamma = \delta$.

Proof. If C is projective, then we are immediately done by the snake lemma. Moreover, the construction of the isomorphism is explicit – it is given by the connecting homomorphism. In general, choose an epimorphism $Q \rightarrow D$ with Q projective and lift the identity map on D to obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega D & \longrightarrow & Q & \longrightarrow & D \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \longrightarrow 0
\end{array}$$

Tensoring it with the short exact sequence $0 \rightarrow F \xrightarrow{\alpha} P \xrightarrow{\beta} A \rightarrow 0$, we have a spatial commutative diagram with bottom face

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Tor}_1(A, D) & \longrightarrow & A \otimes \Omega D & \longrightarrow & A \otimes Q \longrightarrow A \otimes D \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
\text{Tor}_1(A, C) & \longrightarrow & \text{Tor}_1(A, D) & \longrightarrow & A \otimes B & \longrightarrow & A \otimes C \longrightarrow A \otimes D \longrightarrow 0
\end{array}$$

Its front face is the diagram (3.13), and as we just observed, the lemma is true for the back face. The desired result now follows from the naturality of the connecting homomorphism and a trivial diagram chase. \square

Now we can start building structure maps

$$\Delta_i : S^1 \text{Tor}_1(\Omega^{i+1} A, _)(\Sigma^{i+1} B) \longrightarrow S^1 \text{Tor}_1(\Omega^i A, _)(\Sigma^i B).$$

Clearly, it suffices to do this for $i = 0$, and we shall again use the diagram (3.3). This yields a commutative diagram of solid arrows

$$\begin{array}{ccccccc}
 & & \text{Tor}_1(\Omega A, I^1) & & 0 & & \\
 & & \searrow \epsilon_1 & & \downarrow & & \\
 & & \text{Tor}_1(\Omega A, \Sigma^2 B) & \twoheadrightarrow & \text{Tor}_1(A, \Sigma B) & & \\
 & & \searrow \alpha_1 & & \downarrow & & \\
 \Omega A \otimes B & \longrightarrow & \Omega A \otimes I^0 & \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \searrow \\
 0 \longrightarrow & P_0 \otimes B & \longrightarrow & P_0 \otimes I^0 & \longrightarrow & P_0 \otimes \Sigma B & \longrightarrow \Omega A \otimes I^1 \\
 \downarrow & \downarrow & & \downarrow & & \downarrow & \searrow \\
 \text{Tor}_1(A, I^0) \xrightarrow{\epsilon} \text{Tor}_1(A, \Sigma B) & \xrightarrow{\alpha} & A \otimes B & \longrightarrow & A \otimes I^0 & \longrightarrow & A \otimes \Sigma B & \longrightarrow P_0 \otimes I^1 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\
 & & 0 & & 0 & & 0 &
 \end{array}$$

with exact rows and columns. Moreover, the diagonal is a fragment of a long homology exact sequence and, at the same time, the bottom row of (3.3) with the fixed argument specialized to ΩA and with ΣB replaced by $\Sigma^2 B$. The diagram shows that α_1 factors through $\text{Tor}_1(A, \Sigma B)$, giving rise to a unique dotted map making a commutative triangle. As $\text{Tor}_1(A, \Sigma B) \rightarrow \Omega A \otimes \Sigma B$ is monic, the dotted map composed with ϵ_1 is zero, and therefore gives rise to a unique map

$$\text{Coker } \epsilon_1 = S^1 \text{Tor}_1(\Omega A, _)(\Sigma B) \rightarrow \text{Tor}_1(A, \Sigma B)$$

Composing it with the isomorphism $\gamma : \text{Tor}_1(A, \Sigma B) \rightarrow \text{Tor}_1(A, \Sigma B)$ constructed in Lemma 3.3.1 and with the canonical epimorphism

$$\text{Tor}_1(A, \Sigma B) \rightarrow \text{Coker } \epsilon = S^1 \text{Tor}_1(A, _)(B),$$

we declare the resulting composition to be the structure map

$$\Delta_1 : S^1 \text{Tor}_1(\Omega A, _)(\Sigma B) \rightarrow S^1 \text{Tor}_1(A, _)(B).$$

By Lemma 3.3.1, it is compatible with the isomorphisms of [6, Proposition 8.3] (for fixed A and ΩA). Similar arguments yield maps Δ_i for all natural i . We have thus proved

Theorem 3.3.2. *The connecting homomorphism in the diagram (3.3) induces a functor isomorphism of directed systems*

$$(S^1 \text{Tor}_1(\Omega^i A, _) \circ \Sigma^i(_, \Delta_i) \simeq (\Omega^i A \otimes \Sigma^i(_, \Delta_i).$$

This isomorphism is natural in A .

□

In summary, all three constructions of the asymptotic stabilization of the tensor product yield isomorphic results. In particular,

Corollary 3.3.3. *The asymptotic stabilization $\text{T}(A, _)$ and the J -completion of $\text{Tor}(A, _)$ are isomorphic as connected sequences of functors.*

Proof. This follows from the fact that the directed system involved in Triulzi's construction of the J -completion and the directed system used in the construction of the asymptotic stabilization are isomorphic. The isomorphism is precisely that appearing in Theorem 3.3.2 \square

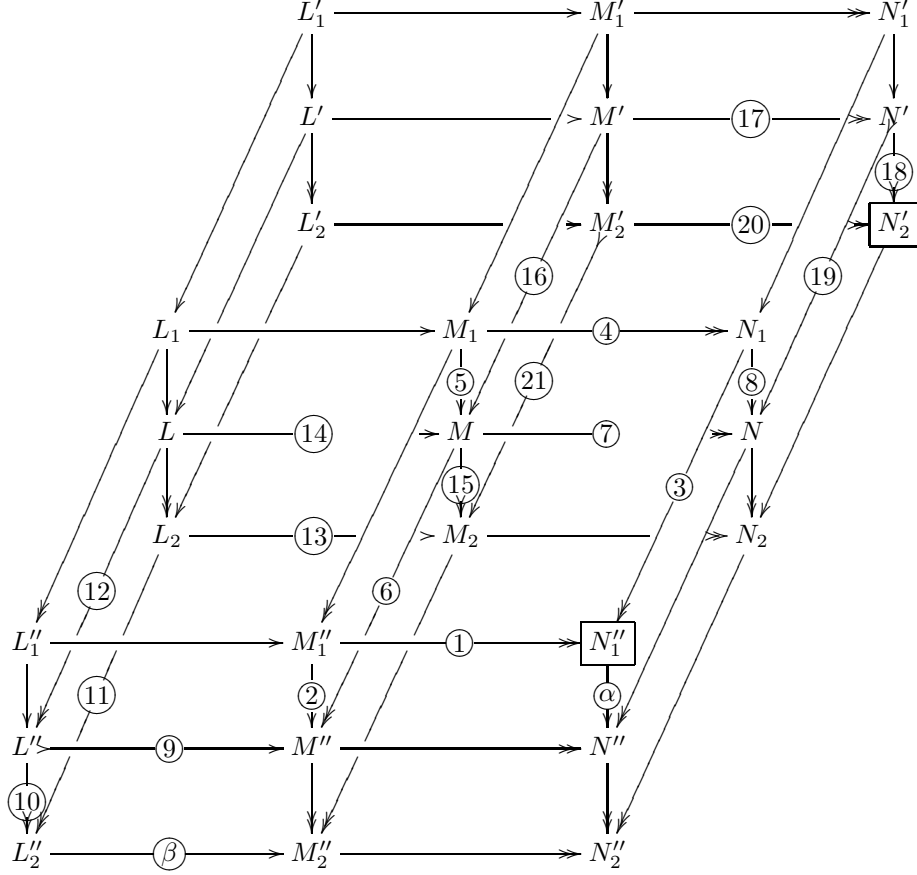
4. TWO LEMMAS ON CONNECTING HOMOMORPHISMS

In this section we shall establish two results, stated and proved in a greater generality than is needed for this paper, helping us understand how to compose connecting homomorphisms running in spatial diagrams. For an exact 3×3 square and a connected sequence of covariant functors, the two possible compositions of the connecting homomorphisms, as in ([3, Proposition 4.1]), always anticommute. In our situation, we do not have a connected sequence of functors; instead, we postulate some properties of the requisite spatial diagrams. The result, however, is similar – the two possible compositions of the connecting homomorphisms anticommute. Our proof, while somewhat tedious, is done by diagram chase and is thus elementary.

We continue to assume that a connecting homomorphism in the snake lemma is constructed by pushing and pulling the elements along a staircase path, as in the traditional proof of the lemma.

4.1. Front, bottom, and right-hand faces.

Lemma 4.1.1. *Let*



be a commutative diagram subject to the following conditions:

- (1) any three-term sequence with arrows running in the same direction is exact (i.e., exact at the middle term);
- (2) each arrow preceded by an arrow in the same direction is epic;
- (3) the three middle three-term sequences $L''M''N''$, $M'_2M_2M''_2$, and $N'N''N''$ on the front, the bottom and the right-hand faces of this cube are short-exact, i.e., each sequence is exact in the middle, the first map is monic, and the second map is epic.

Then the image of the connecting homomorphism $\text{Ker } \alpha \rightarrow L''_2$ (in the front face) is in $\text{Ker } \beta$, and the composition of the connecting homomorphisms $\text{Ker } \alpha \rightarrow L''_2$ and $\text{Ker } \beta \rightarrow N'_2$ (in the bottom face) equals the negative of the connecting homomorphism $\text{Ker } \alpha \rightarrow N'_2$ (in the right-hand face).

Proof. First of all, because of the assumptions on the three short exact sequences, the connecting homomorphisms mentioned in the statement are indeed defined. The first assertion is now immediate. To prove the second assertion, pick an element $n''_1 \in \text{Ker } \alpha \subset N''_1$. By the commutativity of the diagram,

$$m'' := 2 \circ 1^{-1}(n''_1) = 6 \circ 5 \circ 4^{-1} \circ 3^{-1}(n''_1).$$

Along the way, we have an element $m \in M$ such that

$$7(m) = 8 \circ 3^{-1}(n_1'') =: n.$$

Let

$$\mu := (14) \circ (12)^{-1} \circ 9^{-1} \circ 6(m).$$

Since $15(m) = 0$, we have

$$m_2 := 15(\mu - m) = (13) \circ (11)^{-1} \circ (10) \circ 9^{-1}(m'')$$

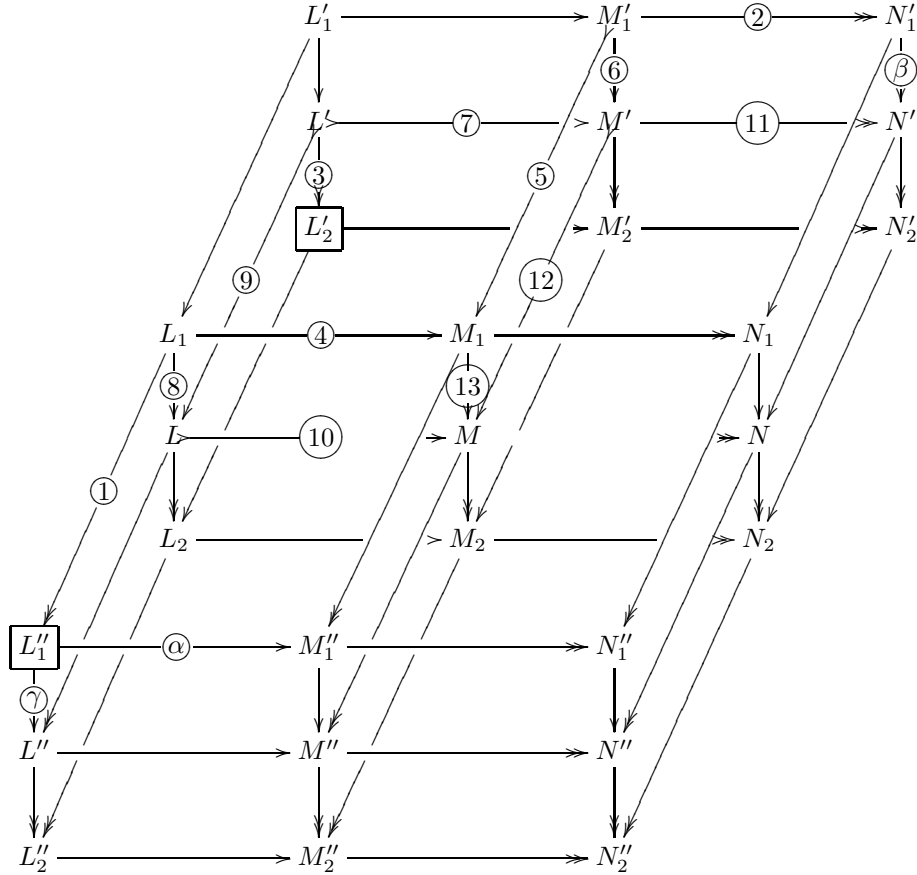
Since $7(\mu) = 0$, we have

$$7(\mu - m) = -8 \circ 3^{-1}(n_1'').$$

By construction, $6(\mu) = 6(m)$, and therefore $\mu - m = 16(m')$ for some $m' \in M'$. Now $(19) \circ (17)(m') = -n$. Therefore, $(18) \circ (17)(m')$ is negative the value of the connecting homomorphism $\text{Ker } \alpha \rightarrow N_2'$ on n_1'' . Using the commutativity of the diagram again, we have the same value for $(20) \circ (21)^{-1}(m_2)$, which is the value of the composition of the other two connecting homomorphisms on the same element. \square

4.2. Top, back, and left-hand faces. Now we look at the composition of connecting homomorphisms in the three remaining planes of the cube.

Lemma 4.2.1. *Let*



be a commutative diagram subject to the following conditions:

- (1) any three-term sequence with arrows running in the same direction is exact;
- (2) each arrow preceded by an arrow in the same direction is epic;
- (3) the three middle three-term sequences $M'_1 M_1 M''_1$, $L' M' N'$, and $L' L L''$ on the top, the back, and the left-hand faces of this cube are short-exact, i.e., each sequence is exact in the middle, the first map is monic, and the second map is epic. Moreover, the two horizontal sequences LMN and $M' M M''$ passing through the center of the cube are also short-exact.

Then the image of $\text{Ker } \alpha \cap \text{Ker } \gamma$ under the connecting homomorphism $\text{Ker } \alpha \rightarrow N'_1$ (in the top face) is in $\text{Ker } \beta$, and on $\text{Ker } \alpha \cap \text{Ker } \gamma$ the composition of the connecting homomorphisms $\text{Ker } \alpha \rightarrow N'_1$ and $\text{Ker } \beta \rightarrow L'_2$ (in the back face) coincides with the connecting homomorphism $\text{Ker } \gamma \rightarrow L'_2$ (in the left-hand face).

Proof. First, we need to show that the connecting homomorphism $\text{Ker } \alpha \rightarrow N'_1$ maps $\text{Ker } \alpha \cap \text{Ker } \gamma$ to $\text{Ker } \beta$. Pick an element $l'' \in \text{Ker } \alpha \cap \text{Ker } \gamma \subset L''_1$ and let

$$m' := 6 \circ 5^{-1} \circ 4 \circ 1^{-1}(l''_1).$$

We need to show that $11(m') = 0$ or, equivalently, that m' is in the image of 7. Let $l := 8 \circ 1^{-1}(l''_1)$. By the commutativity of the diagram,

$$12(m') = 10(l).$$

On the other hand, since $l''_1 \in \text{Ker } \gamma$, $l = 9(l')$ for some $l' \in L'$, and therefore

$$12 \circ 7(l') = 10(l) = 12(m').$$

Since 12 is assumed to be a monomorphism, $m' = 7(l')$, which is the desired claim.

Now we can prove the second claim. Because each of the morphisms 1, 2, and 3 belongs to two connecting homomorphisms, it suffices to show that

$$7^{-1} \circ 6 \circ 5^{-1} \circ 4 = 9^{-1} \circ 8.$$

But the two morphisms become equal when we precompose them with the monomorphism $10 \circ 9$. \square

5. THE ASYMPTOTIC STABILIZATION AS A CONNECTED SEQUENCE OF FUNCTORS

Our next goal is to define, for each short exact sequence

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0,$$

of left Λ modules, connecting homomorphisms

$$\omega_n : T_n(A, B'') \rightarrow T_{n-1}(A, B')$$

and show that $(T_\bullet(A, _), \omega_\bullet)$ is a connected sequence of functors.⁴

We continue to assume that the connecting homomorphism in the snake lemma is defined by pushing and pulling elements along a staircase pattern, as in the standard proof of the lemma.

⁴Any sequence of additive functors can be made connected by choosing the zero map as the connecting homomorphism. Our choice will be nonzero.

5.1. **The first construction.** By Lemma 3.1.5, it suffices to define

$$\omega_1 : T_1(A, B'') \longrightarrow T_0(A, B').$$

To this end, we use the horse-shoe lemma and construct a commutative diagram

$$(5.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B' & \longrightarrow & I^{0'} & \longrightarrow & \Sigma B' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \gamma \\ 0 & \longrightarrow & B & \longrightarrow & I^0 & \longrightarrow & \Sigma B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B'' & \longrightarrow & I^{0''} & \longrightarrow & \Sigma B'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where the rows and columns are exact, and the middle column is a split-exact sequence of injective modules. We will also need an embedding $0 \longrightarrow \Sigma B' \xrightarrow{\epsilon} I^{1'}$ of $\Sigma B'$ into the next step of an injective resolution of B' .

Tensoring this diagram with ΩA gives us another commutative diagram of solid arrows

$$(5.2) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \Omega A \otimes \overline{\otimes} B'' & & \\ & & & & \downarrow & & \\ & & & & \Omega A \otimes B' & \longrightarrow & \Omega A \otimes B \longrightarrow \Omega A \otimes B'' \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega A \otimes I^{0'} & \longrightarrow & \Omega A \otimes I^0 & \longrightarrow & \Omega A \otimes I^{0''} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \Omega A \otimes \overline{\otimes} \Sigma B' & \searrow & \Omega A \otimes \Sigma B' & \xrightarrow{1 \otimes \gamma} & \Omega A \otimes \Sigma B & \longrightarrow & \Omega A \otimes \Sigma B'' \longrightarrow 0 \\ & & \downarrow & \searrow 1 \otimes \epsilon & \downarrow & & \downarrow \\ & & 0 & & \Omega A \otimes I^{1'} & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

with exact rows, columns, and the lower left diagonal. The snake lemma yields a map

$$\kappa : \Omega A \otimes \overline{\otimes} B'' \longrightarrow \Omega A \otimes \Sigma B'$$

with $(1 \otimes \gamma) \circ \kappa = 0$. On the other hand, since $I^{1'}$ is injective, ϵ extends over γ and therefore $1 \otimes \epsilon$ extends over $1 \otimes \gamma$. Hence $(1 \otimes \epsilon) \circ \kappa = 0$, and κ factors through $\text{Ker}(1 \otimes \epsilon) = \Omega A \otimes \Sigma B'$. We have thus constructed a map

$$\kappa_1^1 : \Omega A \otimes B'' \longrightarrow \Omega A \otimes \Sigma B'$$

The same procedure yields maps

$$\kappa_1^i : \Omega^i A \otimes \Sigma^{i-1} B'' \longrightarrow \Omega^i A \otimes \Sigma^i B'$$

for each natural i .

The next step is to show that the maps κ_1^i are compatible with the structure maps Δ . Actually, as we will see, this is not true since the requisite squares anti-commute rather than commute. This motivates

Definition 5.1.1. *For each integer i , set $\omega_1^i := (-1)^i \kappa_1^i$.*

Notice that both the Δ and the κ are connecting homomorphisms in suitable diagrams.

Lemma 5.1.2. *Under the above assumptions and notation, the diagram*

$$\begin{array}{ccc} \Omega^2 A \otimes \Sigma B'' & \xrightarrow{\omega_1^2} & \Omega^2 A \otimes \Sigma^2 B' \\ \downarrow \Delta_1 & & \downarrow \Delta_2 \\ \Omega A \otimes B'' & \xrightarrow{\omega_1^1} & \Omega A \otimes \Sigma B' \end{array}$$

commutes.

Proof. Tensoring the commutative diagram (5.1) with the exact sequence

$$0 \longrightarrow \Omega^2 A \longrightarrow P_1 \longrightarrow \Omega^1 A \longrightarrow 0,$$

where P_1 is a projective module, we have a spatial commutative diagram satisfying all conditions of Lemma 4.1.1. In that diagram, the connecting homomorphism on the front face equals Δ_1 , and the connecting homomorphism on the bottom face equals $\kappa_1^1 = -\omega_1^1$. By the lemma, the composition $\omega_1^1 \Delta_1$ equals the connecting homomorphism on the right-hand vertical face.

Now we shift all indices in the diagram (5.1) one step up and again tensor it with the above short exact sequence. The resulting spatial diagram satisfies all conditions of Lemma 4.2.1. In that diagram, the connecting homomorphism on the top is $\kappa_1^2 = \omega_1^2$, and the connecting homomorphism on the back equals Δ_2 . By the lemma, the composition $\Delta_2 \omega_1^2$ equals the connecting homomorphism on the left-hand vertical face. Since that face coincides with the right-hand vertical face of the former diagram, we have $\omega_1^1 \Delta_1 = \Delta_2 \omega_1^2$. \square

Applying the foregoing lemma repeatedly and passing to the limit, we have the desired homomorphism

$$\omega_1 : T_1(A, B'') \longrightarrow T_0(A, B').$$

As we observed before, the same construction yields homomorphisms

$$\omega_n : T_n(A, B'') \longrightarrow T_{n-1}(A, B').$$

for all integers n .

Theorem 5.1.3. *The pair $(T_\bullet(A, _), \omega_\bullet)$, is a connected sequence of functors.*

Proof. We already remarked that the asymptotic stabilization of the tensor product is an additive functor. Therefore, given an exact sequence of left Λ -modules

$$0 \longrightarrow B' \xrightarrow{\alpha} B \xrightarrow{\beta} B'' \longrightarrow 0,$$

we have that the composition

$$T_n(A, B') \xrightarrow{T_n(A, \alpha)} T_n(A, B) \xrightarrow{T_n(A, \beta)} T_n(A, B'').$$

of the induced maps is zero. The fact that $T_{n-1}(A, \alpha) \circ \omega_n = 0$ follows from the snake lemma applied to the diagram (5.2). For the same reason, $\omega_n \circ T_n(A, \beta) = 0$.

Thus it remains to show that the ω_n are functorial. But this follows from the functoriality of the connecting homomorphism in the snake lemma applied to the diagram (5.2). \square

5.2. The second construction. We continue to work with the right Λ -module A and the short exact sequence

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

of left Λ -modules.

The approach we are about to describe will make use of the functorial long exact sequence [6, (8.1)]. In that sequence, each injective stabilization is part of a functorial directed sequence; this was established in Proposition 3.1.3. In particular, the structure maps Δ_i are functorial in the second argument. This implies that **each row of the injective stabilizations** in the sequence [6, (8.1)] gives rise to morphisms of the requisite directed systems.

Now we want to build structure maps for each of the Tor terms in [6, (8.1)]. In fact, those maps have already been built in the second construction of the asymptotic stabilization, see the diagram (3.12) and the diagram on page 10, where B (respectively, B' , B'') should be replaced with ΣB (respectively, $\Sigma B'$, $\Sigma B''$). As Theorem 3.2.1 shows, the resulting system is functorial in each argument. Therefore, **each row of the Tor functors** in the sequence [6, (8.1)] gives rise to morphisms of the requisite directed systems.

Now we claim that the term-wise maps between the directed systems constitute morphisms of those systems. The foregoing discussion shows that we only have to check the commutativity of the squares lying over the connecting homomorphisms in the sequence [6, (8.1)]. There are three types of such homomorphisms: between two copies of Tor, between Tor and the injective stabilization, and between two copies of injective stabilization. It is clear that we only have to check one square of each type.

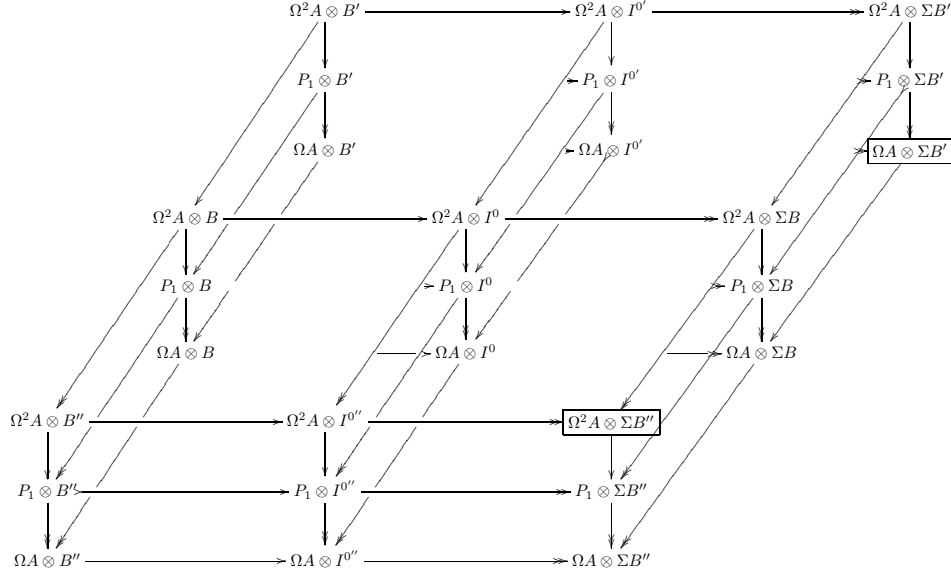
We first examine the square(s) connecting Tor_1 and the requisite injective stabilization.

Lemma 5.2.1. *In the above notation, the square*

$$\begin{array}{ccc} \text{Tor}_1(\Omega A, \Sigma B'') & \xrightarrow{h} & \Omega A \otimes^{\overline{}} \Sigma B' \\ f \downarrow & & \downarrow k \\ \text{Tor}_1(A, B'') & \xrightarrow{g} & A \otimes^{\overline{}} B' \end{array}$$

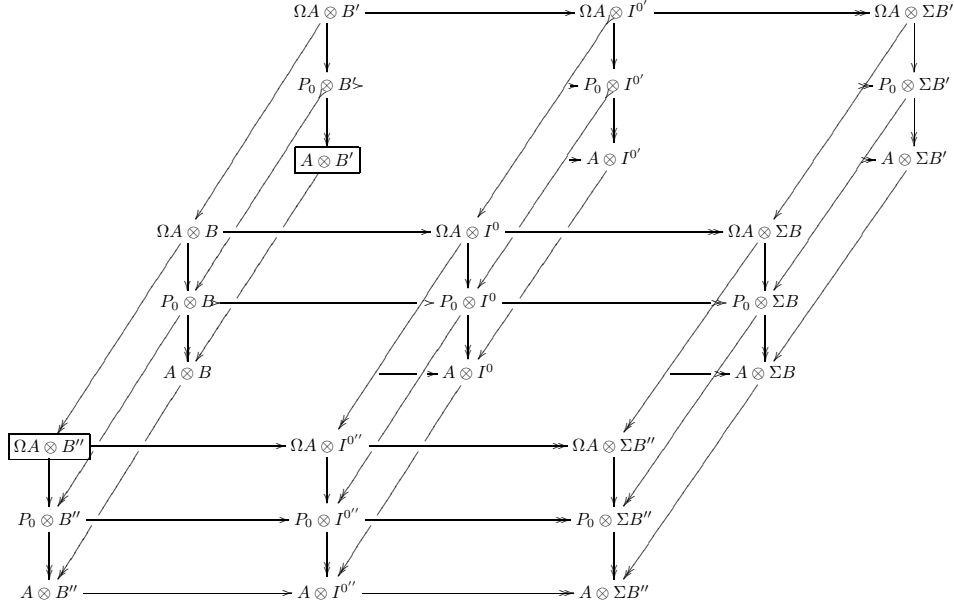
anti-commutes.

Proof. We begin by describing the coinital maps f and h ; they are both connecting homomorphisms in the following commutative 3D diagram



The map f is determined by the front face and starts from inside the framed term, and h is determined by the right-hand face of the cube, with its image inside the framed node in the back.

Let us now describe the coterminal maps g and k . These maps are both connecting homomorphisms in the following commutative 3D diagram



The map g is determined by the left-hand face and starts from inside the framed term, and k is determined by the back face of the cube, with its image inside the framed term in the back.

One can easily check that the first cube satisfies the conditions of Lemma 4.1.1, and the second cube satisfies the conditions of Lemma 4.2.1. However, we cannot immediately apply these results because neither gf nor kh is contained in a single cube. To bypass this obstacle, notice that the bottom face of the first cube coincides with the top face of the second cube. Let δ be the connecting homomorphism in this common face (it starts inside $\Omega A \otimes B''$ and ends inside $\Omega A \otimes \Sigma B'$). By Lemma 4.1.1, $-h = \delta f$ and, by Lemma 4.2.1, $g = k\delta$. Therefore $gf = -kh$, as claimed. \square

Now we look at the square(s) connecting two consecutive Tor functors.

Lemma 5.2.2. *In the above notation, the square*

$$\begin{array}{ccc} \mathrm{Tor}_2(\Omega A, \Sigma B'') & \xrightarrow{h} & \mathrm{Tor}_1(\Omega A, \Sigma B') \\ f \downarrow & & \downarrow k \\ \mathrm{Tor}_2(A, B'') & \xrightarrow{g} & \mathrm{Tor}_1(A, B') \end{array}$$

anti-commutes.

Proof. The argument in this case is identical to that of the previous lemma, except that, in the two cubes, A has to be replaced by ΩA . The details are left to the reader. \square

Finally, we examine the square(s) connecting two consecutive shifts of the injective stabilizations.

Lemma 5.2.3. *In the above notation, the square*

$$\begin{array}{ccc} \Omega A \overline{\otimes} \Sigma B'' & \xrightarrow{h} & \Omega A \overline{\otimes} \Sigma^2 B' \\ f \downarrow & & \downarrow k \\ A \overline{\otimes} B'' & \xrightarrow{g} & A \overline{\otimes} B' \end{array}$$

anti-commutes.

Proof. The argument is similar to those in the previous two lemmas. To describe the composition gf we use the first cube from the proof of Lemma 5.2.1, where we lower the index of each copy of Ω by one. Let δ be the connecting homomorphism in the right-hand face of that cube. By Lemma 4.1.1, $gf = -\delta$.

To describe the composition kh we use the second cube from the proof of Lemma 5.2.1, where we raise the index of each copy of Σ by one (in particular, B becomes ΣB , etc). The left-hand face of this cube coincides with the right-hand face of the previous cube, so they share the connecting homomorphism δ . By Lemma 4.2.1, $\delta = kh$, and therefore $gf = -kh$, as claimed. \square

The just proved results show that, to obtain morphisms of the directed systems, we have to offset the sign in the squares that contain connecting homomorphisms. For example, we can leave the vertical directed systems unchanged, but modify the horizontal long exact sequences by introducing an alternating (in the vertical direction) sign for the connecting homomorphisms. In summary, we have

Theorem 5.2.4. *The pair $(T_\bullet(A, _), \rho_\bullet)$, where ρ_\bullet is the limit of the connecting homomorphisms modified as above, is a connected sequence of functors.*

6. COMPARISON HOMOMORPHISMS

At the moment we have three constructions of stable homology: Vogel's, Triulzi's, and the asymptotic stabilization of the tensor product. Our next goal is to compare them.

6.1. Comparing Vogel homology with the asymptotic stabilization. We want to construct a natural transformation from Vogel homology to the asymptotic stabilization of the tensor product. This will be done in degree zero; all other degrees are treated similarly. Let A be a right Λ -module with a projective resolution $(P, \partial_P) \rightarrow A$, and B be a left Λ -module with an injective resolution $B \rightarrow (I, \partial_I)$. Recall that the differential on $V_\bullet(A, B)$ is induced by $\partial_P \otimes 1 + (-1)^{\deg_1(_)} 1 \otimes \partial_I$ (see (2.1)). To simplify notation, we set

$$d_P := \partial_P \otimes 1 \quad \text{and} \quad d_I := 1 \otimes \partial_I.$$

A homology class in $V_0(A, B)$ can be represented by an infinite sequence

$$s = (s_i)_{i=1}^\infty \in (P_1 \otimes I^0) \times (P_2 \otimes I^1) \times \cdots$$

which vanishes under the differential of $V_\bullet(A, B)$. This means that

$$D(s) = (d_P(s_1), -d_I(s_1) + d_P(s_2), d_I(s_2) + d_P(s_3), -d_I(s_3) + d_P(s_4), \dots)$$

represents the zero class in $V_{-1}(A, B)$ and therefore has only finitely many nonzero components. Let k be the smallest index such that

$$\begin{aligned} d_I(s_k) &= d_P(s_{k+1}) \\ d_I(s_{k+1}) &= -d_P(s_{k+2}) \\ d_I(s_{k+2}) &= d_P(s_{k+3}) \\ d_I(s_{k+3}) &= -d_P(s_{k+4}) \\ &\vdots \end{aligned} \tag{6.1}$$

In short, for all $i \geq 0$

$$d_I(s_{k+i}) = (-1)^i d_P(s_{k+i+1})$$

Observe that since $s_{k+1} \in P_{k+1} \otimes I^k$, $d_P(s_{k+1}) \in P_k \otimes I^k$. Denote $d_P(s_{k+1})$ by \bullet in the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} \Omega^{k+1}A \otimes \Sigma^k B & \longrightarrow & \Omega^{k+1}A \otimes I^k & \longrightarrow & \Omega^{k+1}A \otimes \Sigma^{k+1}B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & P_k \otimes \Sigma^k B & \longrightarrow & P_k \otimes I^k & \longrightarrow & P_k \otimes \Sigma^{k+1}B & \longrightarrow 0 \\ & \downarrow \square & & \downarrow \bullet & & \downarrow & \\ \Omega^k A \otimes \Sigma^k B & \longrightarrow & \Omega^k A \otimes I^k & \longrightarrow & \Omega^k A \otimes \Sigma^{k+1}B & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

Since $\bullet = d_I(s_k)$, pulls back to some element \square . Pushing it down, we produce $\omega_k \in \Omega^k A \otimes \Sigma^k B$. Since $\bullet = d_P(s_{k+1})$, the element \bullet is the image of some element \circ in $\Omega^{k+1} A \otimes I^k$. By the commutativity of the diagram, the image of ω_k in $\Omega^k A \otimes I^k$ is zero, i.e., $\omega_k \in \Omega^k A \otimes \Sigma^k B$, and we set $\varphi_k := \omega_k$.

This process is well defined up to choice of sign. To see these, notice that the element \bullet goes to 0 when applying the horizontal map. Hence by commutativity of the diagram s_{k+1} also goes to 0 using the vertical top right map and hence is in the kernel of this map. Now one can apply the map from the snake lemma to produce the exact same element ω_k . Since we may also take the negation of this connecting homomorphism, we even have the freedom to choose $\pm \omega_k$. Once this choice is fixed, ω_k will be well defined.

Next, apply the same procedure to $d_P(s_{k+2}) = -d_I(s_{k+1}) \in P_{k+1} \otimes I^{k+1}$, producing $\omega_{k+1} \in \Omega^{k+1} A \otimes \Sigma^{k+1} B$. Flipping its sign, we set $\varphi_{k+1} := -\omega_{k+1}$. To obtain φ_{k+2} , perform the same procedure with $d_P(s_{k+3})$ and set $\varphi_{k+2} := -\omega_{k+2}$. To obtain φ_{k+3} , perform the same procedure with $d_P(s_{k+4})$ and set $\varphi_{k+3} := \omega_{k+3}$. Iterating this process, for any $i \geq 0$, we set

$$\varphi_{k+i} := \begin{cases} \omega_{k+i} & \text{if } i \equiv 0, 3 \pmod{4} \\ -\omega_{k+i} & \text{if } i \equiv 1, 2 \pmod{4}. \end{cases}$$

We claim that the sequence $(\varphi_k, \varphi_{k+1}, \dots)$ is coherent, i.e., in the notation of (3.4), $\Delta_n(\varphi_n) = \varphi_{n-1}$ for any $n \geq k+1$. It suffices to check this claim for $n = k+1$; the remaining cases are similar. To this end, we examine the commutative diagram

$$\begin{array}{ccccc}
 & & & \xrightarrow{d_I} & \\
 & & P_{k+1} \otimes I^k & \xrightarrow{\textcircled{5}} & P_{k+1} \otimes \Sigma^{k+1} B & \xrightarrow{\textcircled{1}} & P_{k+1} \otimes I^{k+1} \\
 & & \downarrow \textcircled{6} & \textcircled{T} & \downarrow \textcircled{2} & & \downarrow \bullet \\
 & & \Omega^{k+1} A \otimes I^k & \xrightarrow{\textcircled{7}} & \Omega^{k+1} A \otimes \Sigma^{k+1} B & \xrightarrow{\varphi_{k+1}} & \Omega^{k+1} A \otimes I^{k+1} \\
 & & \downarrow \textcircled{8} & & \downarrow & & \\
 P_k \otimes \Sigma^k B & \xrightarrow{\textcircled{3}} & P_k \otimes I^k & \xrightarrow{\bullet} & P_k \otimes \Sigma^{k+1} B & & \\
 \downarrow \textcircled{4} & & & & & & \\
 \Omega^k A \otimes \Sigma^k B & & & & & & \\
 \varphi_k & & & & & &
 \end{array}$$

Here $(1) \circ (5) = d_I$, $(8) \circ (6) = d_P$, and the bullets denote

$$d_P(s_{k+2}) = -d_I(s_{k+1}) = -(1) \circ (5)(s_{k+1}) \quad \text{and} \quad d_P(s_{k+1}).$$

The element φ_{k+1} is obtained from the upper bullet by applying $(2) \circ (1)^{-1}$ and φ_k is obtained from the lower bullet by applying $(4) \circ (3)^{-1}$. Since the square T commutes,

$$\varphi_{k+1} = -(2) \circ (1)^{-1}(d_P(s_{k+2})) = -(7) \circ (6) \circ (5)^{-1} \circ (1)^{-1}(d_P(s_{k+2})).$$

On the other hand, recalling the construction of Δ_{k+1} (this is just the restriction of the connecting homomorphism in the diagram (3.3)) we have

$$\begin{aligned}
\Delta_{k+1}(\varphi_{k+1}) &= (4) \circ (3)^{-1} \circ (8) \circ (7)^{-1}(\varphi_{k+1}) \\
&= -(4) \circ (3)^{-1} \circ (8) \circ (7)^{-1} \circ (7) \circ (6) \circ (5)^{-1} \circ (1)^{-1}(d_P(s_{k+2})) \\
&= (4) \circ (3)^{-1} \circ (8) \circ (6) \circ (5)^{-1} \circ (1)^{-1}(d_I(s_{k+1})) \\
&= (4) \circ (3)^{-1} \circ (8) \circ (6)(s_{k+1}) \\
&= (4) \circ (3)^{-1}(d_P(s_{k+1})) \\
&= \varphi_k.
\end{aligned}$$

Thus we have shown that the sequence $(\varphi_k, \varphi_{k+1}, \dots)$ is coherent. It uniquely extends to a coherent sequence $(\varphi_i)_{i=0}^\infty$, and we set $\kappa_0(s) := (\varphi_i)_{i=0}^\infty$. A similar argument yields $\kappa_l : V_l(A, -) \longrightarrow T_l(A, -)$ for each integer l .

Theorem 6.1.1. *Let A be a right Λ -module. For each $l \in \mathbb{Z}$,*

$$\kappa_l : V_l(A, -) \longrightarrow T_l(A, -)$$

is a natural transformation.

Proof. We only need to show the naturality of each κ_l . But this follows from the naturality of the connecting homomorphism. \square

Theorem 6.1.2. *In the above notation, for each $l \in \mathbb{Z}$, the natural transformation $\kappa_l : V_l(A, -) \longrightarrow T_l(A, -)$ is an epimorphism.*

Proof. The proof is primarily diagram chase and only a sketch will be given. Let $(\varphi_0, \varphi_1, \varphi_2, \dots)$ be a coherent sequence in the asymptotic stabilization of the tensor product. Then $\varphi_k \in \Omega^k A \otimes \Sigma^k B$. We will construct an element in $V_0(A, B)$ which

maps onto this coherent sequence. One will benefit from the following diagram:

$$\begin{array}{ccccccc}
 & & P_2 \otimes I^0 & \longrightarrow & P_2 \otimes I^1 & \longrightarrow & P_2 \otimes \Sigma^2 B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Omega^2 A \otimes \Sigma^1 B & \dashrightarrow & \Omega^2 A \otimes I^1 & \dashrightarrow & \Omega^2 A \otimes \Sigma^2 B \dashrightarrow 0 \\
 & & \vdots & & \vdots & & \downarrow \varphi_2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \longrightarrow & P_1 \otimes B & \longrightarrow & P_1 \otimes I^0 & \longrightarrow & P_1 \otimes \Sigma^1 B & \dashrightarrow & P_1 \otimes I^1 & \dashrightarrow & P_1 \otimes \Sigma^2 B \dashrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \Omega^1 A \otimes B & \Longrightarrow & \Omega^1 A \otimes I^0 & \Longrightarrow & \Omega^1 A \otimes \Sigma^1 B & \dashrightarrow & \Omega^1 A \otimes I^1 & \dashrightarrow & \Omega^1 A \otimes \Sigma^2 B \dashrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow \varphi_1 & & \downarrow & & \downarrow \\
 0 \Longrightarrow & P_0 \otimes B & \Longrightarrow & P_0 \otimes I^0 & \Longrightarrow & P_0 \otimes \Sigma^1 B & & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & & \\
 & A \otimes B & \Longrightarrow & A \otimes I^0 & \Longrightarrow & A \otimes \Sigma B & & & & \\
 & \downarrow \varphi_0 & & \downarrow & & \downarrow & & & & \\
 & 0 & & 0 & & 0 & & & &
 \end{array}$$

Start by selecting $s_1 \in P_1 \otimes I^0$ that maps onto φ_1 . Then $d_P(s_1)$ will pullback to φ_0 . Now select $t_2 \in P_2 \otimes I^1$ that maps onto φ_2 . By diagram chase we get that there exists $y_2 \in P_2 \otimes I^0$ such that $d_I(s_1) - d_P(t_2) = d_P(d_I(y_2))$ which yields $d_I(s_1) = d_P(t_2 - d_I(y_2))$. Define $s_2 := t_2 - d_I(y_2)$. Then s_2 still maps onto φ_2 and $d_P(s_2)$ pulls back to φ_1 .

Now select $t_3 \in P_3 \otimes I^2$ that maps onto $-\varphi_3$. Then $-d_P(t_3)$ pulls back to φ_2 as does $d_I(s_2)$. By diagram chasing, there exists $y_3 \in P_3 \otimes I^1$ such that $d_I(s_2) + d_P(t_3) = d_P(d_I(y_3))$. Define $s_3 := t_3 - d_I(y_3)$. Then s_3 maps onto $-\varphi_3$ so $-d_P(s_3)$ pulls back to φ_2 . Moreover $d_I(s_2) = -d_P(s_3)$.

If we continue this process paying close attention to signs, we can construct an element $(s_k)_{k=1}^\infty \in V_0(A, B)$ that maps onto the coherent sequence $(\varphi_k)_{k=1}^\infty$. The details are left to the reader. \square

Let U be a connected sequence of functors and denote by $M_\bullet(U)$ its J -completion (see 2.2). In [9, Proposition 6.1.2], Triulzi shows that there is a morphism of connected sequences of functors $\tau : M_\bullet(U) \rightarrow U$ satisfying the following universal property. Given any morphism $\beta : V \rightarrow U$, where V is a connected sequence of functors that is injectively stable in all degrees, there exists a unique morphism $\phi : V \rightarrow M_\bullet(U)$ such that $\phi\tau = \beta$. From this, we can now establish a commutative diagram of comparison maps between Vogel homology, the asymptotic stabilization of the tensor product, and Triulzi's J -completion of the functor Tor .

Proposition 6.1.3. *For any module A , there is a commutative diagram of connected sequences of functors.*

$$\begin{array}{ccc}
 & V_{\bullet}(A, -) & \\
 \swarrow \kappa & & \searrow \theta \\
 T_{\bullet}(A, -) & \xrightarrow{\simeq} & M_{\bullet}(\text{Tor}(A, -)) \\
 \searrow \lambda & & \swarrow \tau \\
 & \text{Tor}(A, -) &
 \end{array}$$

Proof. The horizontal isomorphism is precisely that appearing in Theorem 3.3.2. The morphism $\lambda\kappa : V_{\bullet}(A, -) \rightarrow \text{Tor}(A, -)$ factors uniquely through τ because the connected sequence of functors $V_{\bullet}(A, -)$ is J -complete, that is injectively stable in every degree. The morphism θ is induced by the universal property of $M_{\bullet}(A, -)$ being the J -completion of $\text{Tor}(A, -)$. \square

Corollary 6.1.4. *The comparison map from Vogel homology to the J -completion of Tor is epic in each degree.*

\square

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